J. Fluid Mech. (1976), vol. 74, part 3, pp. 401–413
Printed in Great Britain

Mass transport in the bottom boundary layer of cnoidal waves

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(Received 9 July 1975)

This study deals with the mass-transport velocity within the bottom boundary layer of cnoidal waves progressing over a smooth horizontal bed. Mass-transport velocity distributions through the boundary layer are derived and compared with that predicted by Longuet-Higgins (1953) for sinusoidal waves. The mass transport at the outer edge of the boundary layer is compared with various theoretical results for an inviscid fluid based on cnoidal wave theory and also with previous experimental results. The effect of the viscous boundary layer is to establish uniquely the bottom mass transport and this is appreciably greater than the somewhat arbitrary prediction for an inviscid fluid.

1. Introduction

In progressive surface waves, the fluid-particle motions include a steady drift, the mass-transport velocity, which was first predicted theoretically by Stokes (1847) for irrotational waves with a sinusoidal first-order motion. A significant advance in understanding mass-transport phenomena was made when Longuet-Higgins (1953) presented a general theory in which the effects of viscosity were included. His results are notably different from those given by Stokes' irrotational theory, and, in particular, they predict the observed forward mass-transport velocity close to the bed.

Nevertheless, significant departures from the theory do occur under various conditions. In recent years, cnoidal waves have received increasing attention, and for shallow-water waves the first-order irrotational motion is perhaps better represented by cnoidal wave theory rather than as being simple harmonic. Le Méhauté (1968) has used the cnoidal theory of Laitone (1960) to calculate the mass transport for cnoidal waves assuming an inviscid fluid, and he has found it to be invariant with depth. Using an alternative approach, Spielvogel & Spielvogel (1974) have derived an expression for the mass transport near the bed, but this does not correspond to Le Méhauté's results. Furthermore, these various expressions are somewhat arbitrary, depending on the definition of wave speed adopted for the irrotational flow.

The purpose of the present paper is to investigate the bottom boundary layer of a cnoidal wave train and so derive the first approximation to the mass-transport velocity distribution close to the bed. The method employed has previously been applied to a first-order sinusoidal motion, summarized for example by Batchelor

(1967, p. 358), and is here extended to the case of cnoidal wave motion outside the boundary layer. The validity of conflicting formulae for the mass transport which have been given previously is discussed and a comparison of the results obtained here with experimental data is also made and indicates reasonable agreement. The effect of the viscous boundary layer is to establish uniquely the bottom mass transport, and this is found to be significantly greater than that expected for an inviscid fluid.

2. Theory of the boundary layer

The waves are assumed to be steady and two-dimensional, and to propagate through an incompressible fluid over a smooth horizontal bed. Only the viscous boundary layer close to the bed is considered, and the first-order motion outside the boundary layer is taken to be given by cnoidal wave theory (Laitone 1960) applied at the bed. Effects of viscous attentuation of the wave motion are consequently neglected. Let x denote the co-ordinate in the direction of wave propagation, y the co-ordinate measured vertically upwards from the bed and u and v the corresponding velocity components. In addition, let t denote time, ν kinematic viscosity, and U the horizontal velocity outside the boundary layer.

It is reasonable to suppose that the boundary layer is thin compared with the characteristic length scales of the wave train, and therefore that the Prandtl boundary-layer equations adequately describe the motion within the boundary layer:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2.1}$$

$$\partial u/\partial x+\partial v/\partial y=0, \tag{2.2}$$

subject to the boundary conditions $u=v=0$ at $y=0$ and $u=U$ as $y\rightarrow\infty$.

Although the vertical velocity V and $\partial U/\partial y$ are both of ordinary magnitude in the fluid's interior, the term $V\partial U/\partial y$ is absent from the right-hand side of (2.1) as part of the boundary-layer approximation. At the boundary-layer edge $(y\rightarrow\infty)$, $\partial U/\partial y$ is taken to have its value at the bed in a wholly irrotational flow, which is zero, and $V\partial U/\partial y$ will in fact be of the order of the other terms neglected.

As discussed by Isaacson & Isaacson (1975, p. 108), dimensional considerations indicate that the problem is one in which physical independence exists between the x and y directions and is one which is dimensionally homogeneous in the extended set of dimensions $MXYT$, where X and Y denote the length dimensions in the x and y directions respectively. We accordingly reduce the problem to one involving variables which are dimensionless in this extended set. The characteristic time and the characteristic lengths in the x and y directions are $1/\omega$, $1/k$ and δ respectively, where ω is the angular wave frequency, k the wavenumber and $\delta=(2\nu/\omega)^{\frac{1}{2}}$ the boundary-layer thickness. (Note that the dimensions of ν in the extended set are Y^2/T.) We therefore put

$$\xi=kx,\quad \tau=\omega t,\quad \eta=y/\delta,\quad u'=ku/\omega,\quad U'=kU/\omega,\quad v'=v/\omega\delta. \tag{2.3}$$

The boundary-layer equations (2.1) and (2.2) may then be written in a non-dimensional form:

$$\left.\begin{aligned}\frac{\partial u'}{\partial \tau}+u'\frac{\partial u'}{\partial \xi}+v'\frac{\partial u'}{\partial \eta}=\frac{\partial U'}{\partial \tau}+U'\frac{\partial U'}{\partial \xi}+\frac{1}{2}\frac{\partial^2 u'}{\partial \eta^2},\\ \partial u'/\partial \xi+\partial v'/\partial \eta=0,\end{aligned}\right\} \tag{2.4}$$

with boundary conditions $u' = v' = 0$ at $\eta = 0$ and $u' = U'$ as $\eta \to \infty$.

A perturbation procedure is now introduced in which it is assumed that u', v' and U' may be expanded as power series in a small parameter ϵ:

$$\left.\begin{aligned}u' &= \epsilon u_1' + \epsilon^2 u_2' + \ldots,\\ v' &= \epsilon v_1' + \epsilon^2 v_2' + \ldots,\\ U' &= \epsilon U_1' + \epsilon^2 U_2' + \ldots.\end{aligned}\right\} \tag{2.5}$$

Substituting (2.5) in (2.4) and collecting powers of ϵ, we obtain at first order

$$\left.\begin{aligned}\frac{\partial u_1'}{\partial \tau}-\frac{1}{2}\frac{\partial^2 u_1'}{\partial \eta^2}=\frac{\partial U_1'}{\partial \tau},\\ \partial u_1'/\partial \xi+\partial v_1'/\partial \eta=0,\end{aligned}\right\} \tag{2.6}$$

with

$$\left.\begin{aligned}u_1' = v_1' = 0 \quad &\text{at} \quad \eta = 0,\\ u_1' = U_1' \quad &\text{as} \quad \eta \to \infty.\end{aligned}\right\} \tag{2.7}$$

At second order, we shall require only the terms independent of time, and with temporal mean values denoted by overbars, we have

$$\overline{\frac{\partial u_2'}{\partial \tau}}-\frac{1}{2}\overline{\frac{\partial^2 u_2'}{\partial \eta^2}}=-\left(\overline{u_1'\frac{\partial u_1'}{\partial \xi}}+\overline{v_1'\frac{\partial u_1'}{\partial \eta}}\right)+\overline{\frac{\partial U_2'}{\partial \tau}}+\overline{U_1'\frac{\partial U_1'}{\partial \xi}}, \tag{2.8}$$

with

$$\overline{u_2'} = 0 \quad \text{at} \quad \eta = 0. \tag{2.9}$$

The motion will be assumed periodic in time and therefore

$$\overline{\partial u_2'/\partial \tau} = [u_2']_{\tau}^{\tau+2\pi} = 0. \tag{2.10}$$

Similarly

$$\overline{\partial U_2'/\partial \tau} = 0, \tag{2.11}$$

and (2.8) then gives for $\overline{u_2'}$

$$-\frac{1}{2}\frac{\partial^2 \overline{u_2'}}{\partial \eta^2}=-\left(\overline{u_1'\frac{\partial u_1'}{\partial \xi}}+\overline{v_1'\frac{\partial u_1'}{\partial \eta}}\right)+\overline{U_1'\frac{\partial U_1'}{\partial \xi}}. \tag{2.12}$$

An additional boundary condition is also necessary to obtain a complete solution for $\overline{u_2'}$, and we take

$$\overline{\partial u_2'/\partial \eta} = 0 \quad \text{as} \quad \eta \to \infty. \tag{2.13}$$

This has been accounted for in physical terms by Batchelor (1967, p. 360).

The mass-transport velocity to second order is given by Longuet-Higgins (1953) as

$$U_M = \epsilon^2\left\{\overline{u_2}+\overline{\frac{\partial u_1}{\partial x}\int^t u_1\,dt'}+\overline{\frac{\partial u_1}{\partial y}\int^t v_1\,dt'}\right\} \tag{2.14}$$

and may be written in terms of the dimensionless variables introduced in (2.3) as

$$\frac{U_M}{\epsilon^2 c} = \overline{u_2'} + \overline{\frac{\partial u_1'}{\partial \xi}\int^{\tau} u_1' d\tau'} + \overline{\frac{\partial u_1'}{\partial \eta}\int^{\tau} v_1' d\tau'}, \tag{2.15}$$

where c is the wave speed ($= \omega/k$). The expression (2.14) was derived under the assumptions that the motion is periodic in time, that U_M, u and v may be expressed as power series in ϵ, and that there are no steady first-order motions. No assumptions are made *a priori* concerning the definition of ϵ, and in accordance with the results of cnoidal theory we shall here put $\epsilon = H/h$, where H is the wave height and h the trough depth. Although cnoidal theory is developed on the basis of an alternative perturbation parameter, its results are expressed as power series in H/h, and equating ϵ to this ratio enables those results to be applied directly to the present problem.

The solutions to (2.6) and (2.12) will lead to the determination of $U_M/\epsilon^2 c$, but before proceeding to derive these, it remains to consider the form of the first-order horizontal velocity U_1 just outside the boundary layer, which is taken from cnoidal theory. This is given by Le Méhauté (1968) in terms of the Jacobian elliptic function cn, which has argument $q = (K(\kappa)/\pi)(kx-\omega t)$ and modulus κ, as

$$U_1 = -\frac{(gh)^{\frac{1}{2}}}{\kappa^2}(\gamma - \kappa'^2 - \kappa^2 \mathrm{cn}^2 q), \tag{2.16}$$

where a minus sign has been introduced because here x is measured in the direction of wave propagation. Also, γ is the ratio $E(\kappa)/K(\kappa)$, g is the gravitational acceleration, $E(\kappa)$ is the complete elliptic integral of the second kind, $K(\kappa)$ is the complete elliptic integral of the first kind and $\kappa'^2 = 1-\kappa^2$.

It will be noted that, since $\overline{\kappa^2 \mathrm{cn}^2 q} = \gamma - \kappa'^2$, the assumption of zero first-order steady motion inherent in the derivation of (2.14) is satisfied and U_1' may be written as a complex Fourier series

$$\left.\begin{aligned} U_1' &= \sum_{n=-\infty}^{\infty} A_n e^{in\theta}, \\ \text{with} \qquad A_{-n} &= A_n^*, \quad A_0 = 0. \end{aligned}\right\} \tag{2.17}$$

The asterisk denotes the complex conjugate and

$$\theta = \xi - \tau = kx - \omega t. \tag{2.18}$$

Now cn(q) may be expanded as a Fourier series (e.g. Abramowitz & Stegun 1965, p. 575):

$$\mathrm{cn}(q) = \frac{2\pi}{\kappa K(\kappa)} \sum_{n=1}^{\infty} \beta_n \cos\left(\frac{n\theta}{2}\right), \tag{2.19}$$

$$\text{where} \qquad \left.\begin{aligned} \beta_n &= \begin{cases} 0 & \text{for} \quad n \text{ even}, \\ r^{\frac{1}{2}n}/(1+r^n) & \text{for} \quad n \text{ odd}, \end{cases} \\ r &= \exp[-\pi K(\kappa')/K(\kappa)]. \end{aligned}\right\} \tag{2.20}$$

Squaring and substituting in (2.16), we obtain

$$U_1 = (gh)^{\frac{1}{2}}\left(\frac{2\pi}{\kappa K(\kappa)}\right)^2 \sum_{n=1}^{\infty} B_n \cos n\theta, \tag{2.21}$$

where

$$B_n = \frac{1}{2}\sum_{m=1}^{2n-1} \beta_m \beta_{2n-m} + \sum_{m=1}^{\infty} \beta_m \beta_{2n+m}. \tag{2.22}$$

If we put $A'_n = [c/(gh)^{\frac{1}{2}}]A_n$ for all n, then from (2.17) and (2.21)

$$\left.\begin{aligned} A'_n &= \tfrac{1}{2}(2\pi/\kappa K(\kappa))^2 B_n \quad \text{for} \quad n \geqslant 1,\\ A'_{-n} &= A'_n, \quad A'_0 = 0. \end{aligned}\right\} \tag{2.23}$$

Hence A'_n depends only on the modulus κ and may be determined numerically for all n for any given value of κ.

We now derive the solution to (2.6) in terms of A_n by representing u'_1 and v'_1 as complex Fourier series:

$$\left.\begin{aligned} u'_1 &= \sum_{n=-\infty}^{\infty} a_n e^{in\theta}, \quad a_{-n} = a_n^*,\\ v'_1 &= \sum_{n=-\infty}^{\infty} b_n e^{in\theta}, \quad b_{-n} = b_n^*. \end{aligned}\right\} \tag{2.24}$$

The solution that satisfies the boundary conditions (2.7) is found to be

$$\left.\begin{aligned} a_n &= A_n[1-\exp(-\alpha_n \eta)],\\ b_n &= -inA_n\left(\eta + \frac{\exp(-\alpha_n\eta)}{\alpha_n} - \frac{1}{\alpha_n}\right), \end{aligned}\right\} \tag{2.25}$$

where

$$\alpha_n = (1-i)\,n^{\frac{1}{2}}. \tag{2.26}$$

Now if F and G are any two periodic functions with zero mean values given by

$$\left.\begin{aligned} F &= \sum_{n=-\infty}^{\infty} f_n e^{in\theta}, \quad f_{-n} = f_n^*, \quad f_0 = 0,\\ G &= \sum_{n=-\infty}^{\infty} g_n e^{in\theta}, \quad g_{-n} = g_n^*, \quad g_0 = 0, \end{aligned}\right\} \tag{2.27}$$

then

$$\overline{FG} = \sum_{n=1}^{\infty} (f_n g_n^* + f_n^* g_n). \tag{2.28}$$

By applying (2.28) to the various terms on the right-hand side of (2.12), we can obtain an expression for the Eulerian drift $\overline{u'_2}$:

$$\overline{u'_1\frac{\partial u'_1}{\partial \xi}} = \overline{U'_1\frac{\partial U'_1}{\partial \xi}} = 0. \tag{2.29}$$

Also

$$\overline{v'_1\frac{\partial u'_1}{\partial \eta}} = \sum_{n=1}^{\infty} inA_n^2\left\{\left(\alpha_n\eta - \frac{\alpha_n}{\alpha_n^*}\right)\exp(-\alpha_n\eta) - \left(\alpha_n\eta - \frac{\alpha_n^*}{\alpha_n}\right)\exp(-\alpha_n^*\eta) + \left(\frac{\alpha_n}{\alpha_n^*} - \frac{\alpha_n^*}{\alpha_n}\right)\exp[-(\alpha_n+\alpha_n^*)\eta]\right\}. \tag{2.30}$$

By substituting (2.29) and (2.30) into the right side of (2.12), integrating

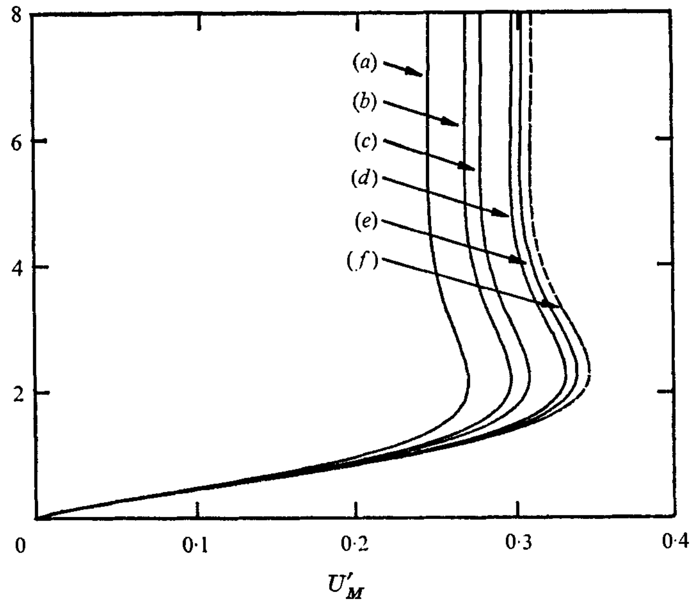

FIGURE 1. Mass-transport velocity distributions through the bottom boundary layer for various values of the modulus κ: (a) κ = 0·999, (b) κ = 0·995, (c) κ = 0·99, (d) κ = 0·95, (e) κ = 0·90, (f) Longuet-Higgins' (1953) solution for shallow-water waves.

twice with respect to η and applying the boundary conditions (2.9) and (2.13), it is found that

$$\overline{u_2'} = \sum_{n=1}^{\infty} 2inA_n^2\left\{\left(\frac{\eta}{\alpha_n}+\frac{2}{\alpha_n^2}-\frac{1}{\alpha_n\alpha_n^*}\right)\exp(-\alpha_n\eta)-\left(\frac{\eta}{\alpha_n^*}+\frac{2}{\alpha_n^{*2}}-\frac{1}{\alpha_n\alpha_n^*}\right)\exp(-\alpha_n^*\eta)\right.$$
$$\left.+\left(\frac{\alpha_n/\alpha_n^*-\alpha_n^*/\alpha_n}{(\alpha_n+\alpha_n^*)^2}\right)\exp[-(\alpha_n+\alpha_n^*)\eta]-\left[\frac{2}{\alpha_n^2}-\frac{2}{\alpha_n^{*2}}+\frac{\alpha_n/\alpha_n^*-\alpha_n^*/\alpha_n}{(\alpha_n+\alpha_n^*)^2}\right]\right\}. \quad (2.31)$$

Equation (2.28) is now re-applied to determine the Lagrangian component of the drift on the right of (2.15):

$$\overline{\frac{\partial u_1'}{\partial \xi}\int^{\tau} u_1'\,d\tau'} = \sum_{n=1}^{\infty} 2A_n^2\{1-\exp(-\alpha_n\eta)-\exp(-\alpha_n^*\eta)+\exp[-(\alpha_n+\alpha_n^*)\eta]\}, \quad (2.32)$$

$$\overline{\frac{\partial u_1'}{\partial \eta}\int^{\tau} v_1'\,d\tau'} = \sum_{n=1}^{\infty} A_n^2\left\{\left(\alpha_n\eta-\frac{\alpha_n}{\alpha_n^*}\right)\exp(-\alpha_n\eta)+\left(\alpha_n^*\eta-\frac{\alpha_n^*}{\alpha_n}\right)\exp(-\alpha_n^*\eta)\right.$$
$$\left.+\left(\frac{\alpha_n}{\alpha_n^*}+\frac{\alpha_n^*}{\alpha_n}\right)\exp[-(\alpha_n+\alpha_n^*)\eta]\right\}. \quad (2.33)$$

The mass-transport velocity as given by (2.15) is now obtained simply by summing the right-hand sides of (2.31), (2.32) and (2.33). Substituting for α_n and A_n, we obtain after some simplification

$$U'_M = \sum_{n=1}^{\infty} A_n'^2\{5+3\exp(-2n^{\frac{1}{2}}\eta)-8\exp(-n^{\frac{1}{2}}\eta)\cos(n^{\frac{1}{2}}\eta)\}, \quad (2.34)$$

where
$$U'_M = U_M/\epsilon^2 ghc^{-1}. \quad (2.35)$$

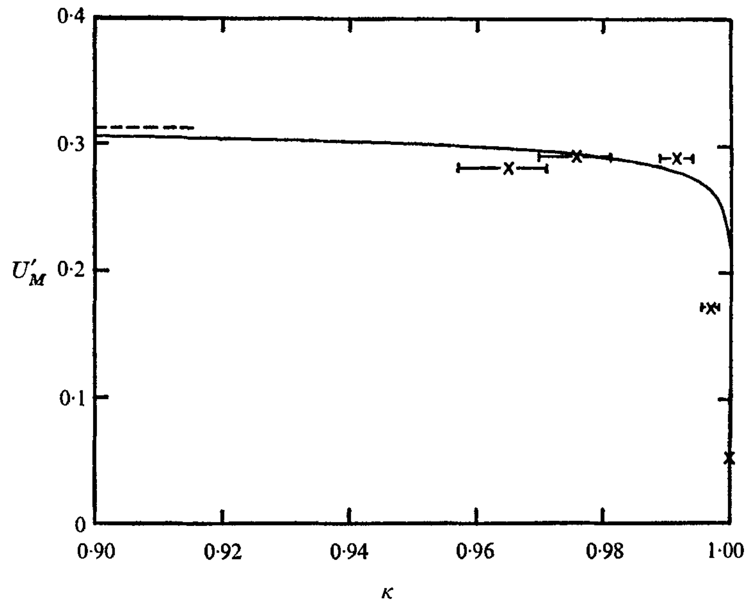

FIGURE 2. The mass-transport velocity at the outer edge of the bottom boundary layer as a function of the modulus κ. – – –, limiting value of U'_M for small κ and Longuet-Higgins' (1953) solution for shallow-water waves; $\vdash\!\times\!\dashv$, experimental data of Allen & Gibson (1959) with indications of the range of error in estimates of κ.

Now since the wave speed c is given by cnoidal theory as $(gh)^{\frac{1}{2}}(1+O[\epsilon])$, the introduction of the factor $c/(gh)^{\frac{1}{2}}$ into (2.34) will modify third- and higher-order terms only. Thus, to the present order of approximation, the alternative dimensionless forms of the mass-transport velocity, $U_M/\epsilon^2(gh)^{\frac{1}{2}}$ and $U_M/\epsilon^2 c$, are also given by the right-hand side of (2.34).

It has been shown that A'_n may be determined numerically for all n for any given value of the modulus κ and therefore the mass-transport velocity may be found from (2.34). Some profiles for various values of κ approaching unity are given in figure 1.

The mass transport at the outer edge of the boundary layer can be found by examining (2.34) as η becomes large. In such a case we have

$$U'_M = \sum_{n=1}^{\infty} 5A'^2_n = \frac{5}{2}\overline{\left(\frac{U_1}{(gh)^{\frac{1}{2}}}\right)^2}. \tag{2.36}$$

Substituting from (2.16),

$$\left.\begin{aligned} U'_M &= (5/2\kappa^4)\{(\gamma-\kappa'^2)^2 - 2(\gamma-\kappa'^2)\overline{\kappa^2\mathrm{cn}^2 q} + \overline{\kappa^4\mathrm{cn}^4 q} \\ &= (5/6\kappa^4)\{2\gamma(2-\kappa^2) - 3\gamma^2 - \kappa'^2\}. \end{aligned}\right\} \tag{2.37}$$

The variation of U'_M with κ is presented in figure 2 for values of κ near unity. By expanding γ as a power series in κ^2, it can be shown that U'_M tends towards the value 0·3125 as κ becomes small, although, of course, the use of cnoidal theory is then invalid.

3. Mass transport in an inviscid fluid

Previous studies relating to mass transport in cnoidal waves have been based on irrotational motion of an inviscid fluid. In such a case the wave speed to second (or higher) order is not unique, and may be defined only after an additional assumption has been made. The two most usual assumptions, originally proposed by Stokes (1847), are, first, that the average horizontal velocity at any given location is zero and second, that the average horizontal momentum over a wavelength is reduced to zero by the addition of a uniform motion. These different assumptions affect the mean Eulerian velocity, and the mass-transport velocity is therefore not unique, but will reflect the assumption chosen. In order to contrast the case of a viscous fluid with this and also to compare it with previous work relating to an inviscid fluid, the mass transport for inviscid motion is now derived for the two cases described.

In the first case, the Eulerian component of mass transport is zero, and also U_1 does not vary with y. In the interior of the fluid, (2.14) then reduces to

$$U_M = \epsilon^2 \overline{\frac{\partial U_1}{\partial x} \int^t U_1 dt'} = \epsilon^2 \frac{\overline{U_1^2}}{c}. \tag{3.1}$$

The mass transport according to the first assumption is thus given by

$$U'_M = (3\kappa^4)^{-1} \{2\gamma(2-\kappa^2) - 3\gamma^2 - \kappa'^2\}. \tag{3.2}$$

For a viscous fluid, the mass transport outside the boundary layer is seen to be 2·5 times greater than this value, which is a result analogous to the case of sinusoidal wave motion.

Under the second assumption, U_2 is given (Le Méhauté 1968) by

$$U_2 = -\frac{(gh)^{\frac{1}{2}}}{4\kappa^2} \left\{ \left[2\kappa'^2 + (7\kappa^2 - 2)\,\mathrm{cn}^2 q - 5\kappa^2 \mathrm{cn}^4 q - \frac{\gamma}{\kappa^2}(4\gamma - 4 + 3\kappa^2) \right] \right.$$
$$\left. + 3\left[\left(\frac{y}{h}\right)^2 - 1 \right] [\kappa'^2 + 2(2\kappa^2 - 1)\,\mathrm{cn}^2 q - 3\kappa^2 \mathrm{cn}^4 q] \right\}, \tag{3.3}$$

where, as in (2.16), a sign change is introduced on account of the reversal in the x direction. The mean value of U_2 simplifies to

$$\overline{U}_2 = -\frac{(gh)^{\frac{1}{2}}}{3\kappa^4} \{2\gamma(2-\kappa^2) - 3\gamma^2 - \kappa'^2\}. \tag{3.4}$$

The Lagrangian drift due to the first-order motion is the same as before, and the mass-transport velocity according to the second assumption is the sum of these two components. As mentioned previously, the factor $c/(gh)^{\frac{1}{2}}$ will modify only higher-order terms and thus, to the present order of approximation, the sum is zero:

$$U'_M = 0. \tag{3.5}$$

4. Comparison with other work

The resemblance of (2.34) to the corresponding equation given by Longuet-Higgins (1953) for sinusoidal waves is hardly surprising since a description of U_1 enters the expression only through the form A_n is to take. Thus, when the first-order motion beyond the boundary layer is simple harmonic, we have from small amplitude wave theory and (2.17)

$$A_n = \begin{cases} kh/4\sinh kd & \text{for} \quad n = 1, \\ 0 & \text{for} \quad n > 1, \end{cases} \tag{4.1}$$

where d is the mean depth. Equation (2.34) then reduces to the Longuet-Higgins (1953) solution:

$$U_M = \frac{H^2\omega k}{16\sinh^2 kd}\{5+3e^{-2\eta}-8e^{-\eta}\cos\eta\}. \tag{4.2}$$

The factor $(d/h)^2$ will alter higher-order terms only, and in the non-dimensional form adopted in (2.35), U'_M for shallow-water waves becomes

$$U'_M = \tfrac{1}{16}\{5+3e^{-2\eta}-8e^{-\eta}\cos\eta\}. \tag{4.3}$$

This profile is indicated by a dashed line in figure 1 and represents the present solution for smaller values of κ. The mass-transport velocity outside the boundary layer is in this case clearly

$$U'_M = 0{\cdot}3125. \tag{4.4}$$

The more general result (2.36) giving the mass transport at the boundary-layer edge does not depend on a particular choice for A_n and so applies to any periodic first-order motion. This result was also obtained by Longuet-Higgins (1958, equation (20)), who showed, moreover, that it is independent of the vertical distribution of viscosity and so may apply to turbulent flow as well.

Spielvogel & Spielvogel (1974) have derived an expression for the mass transport near the bed for cnoidal waves under the assumption of an inviscid fluid. In our notation, their expression is

$$\frac{U_M}{\epsilon^2 c} = \frac{1}{\epsilon^2}\left\{-1+\frac{c}{(gh)^{\frac{1}{2}}}\left[1+\epsilon\left(\frac{1}{2\kappa^2}-\frac{1-\gamma}{\kappa^2}\right)+\epsilon^2\left(\frac{\kappa^4-16\kappa^2+1}{40\kappa^4}+\frac{3(1-\gamma)}{4\kappa^2}\right)\right]\right\}. \tag{4.5}$$

Now Le Méhauté (1968) has given expressions for the wave speed c to second order, corresponding to the assumptions outlined in §3 and deriving from the work of Laitone (1960). These are, respectively,

$$\frac{c}{(gh)^{\frac{1}{2}}} = 1+\epsilon\frac{1-2\gamma}{2\kappa^2}+\epsilon^2\left[\frac{\gamma(4+\kappa^2)}{12\kappa^4}-\frac{3\kappa^4+2\kappa^2+13}{120\kappa^4}\right] \tag{4.6}$$

and

$$\frac{c}{(gh)^{\frac{1}{2}}} = 1+\epsilon\frac{1-2\gamma}{2\kappa^2}+\epsilon^2\left[\frac{\gamma(4\gamma-4+3\kappa^2)}{4\kappa^4}-\frac{\kappa^4+14\kappa^2-9}{40\kappa^4}\right]. \tag{4.7}$$

Substitution of (4.6) and (4.7) in turn into (4.5) and simplification to maintain the present order of approximation gives (3.2) and (3.5) respectively. Thus the results of Spielvogel & Spielvogel (1974) are in agreement with those derived in §3.

Le Méhauté (1968) has also obtained expressions for the mass-transport velocity corresponding to the two cases discussed:

$$U_M/\epsilon^2 c = (24\kappa^4)^{-1}\{5\gamma(2-\kappa^2)+(13\kappa^2-10)\} \tag{4.8}$$

and
$$U_M/\epsilon^2 c = (8\kappa^4)^{-1}\{7\gamma(2-\kappa^2)-8\gamma^2+(7\kappa^2-6)(2-\kappa^2)\}. \tag{4.9}$$

These are in marked contrast to (3.2) and (3.5) respectively. An examination of Le Méhauté's approach indicates that his derivation of the mass-transport velocity is effectively based on the definition

$$U_M = \epsilon^2\left\{\overline{U}_2 + \overline{\frac{\partial U_1}{\partial q_0'} q_1'}\right\}. \tag{4.10}$$

In this context, q' is strictly associated with x', the x co-ordinate of a given particle relative to its position x at some initial time, as well as with the independent variable x itself:

$$q' = \sum_{n=0}^{\infty} \epsilon^n q_n' = \frac{kK(\kappa)}{\pi}(x+x'+ct) \tag{4.11}$$

(with the appropriate sign change). Now k, x' and c are all given as power series in the form

$$f = \sum_{n=0}^{\infty} \epsilon^n f_n, \tag{4.12}$$

with $x_0' = 0$. Therefore

$$q_0' = \frac{k_0 K(\kappa)}{\pi}(x+c_0 t), \quad q_1' = \frac{k_0 K(\kappa)}{\pi}(x_1'+c_1 t) + \frac{k_1}{k_0} q_0'. \tag{4.13}$$

Substitution into (4.10) gives

$$U_M = \epsilon^2\left\{\overline{U}_2 + \overline{\frac{\partial U_1}{\partial x} x_1'} + \left(c_1 + \frac{k_1}{k_0} c_0\right)\overline{\frac{\partial U_1}{\partial x} t}\right\}. \tag{4.14}$$

Only the first two terms on the right side of the above equation correspond to the expression derived by Longuet-Higgins (1953) from first principles and based on the assumptions mentioned earlier. We note that these assumptions do not specify the precise form of ϵ, nor is there any other restriction making that derivation inapplicable to cnoidal waves. The remaining terms on the right side of (4.14) are extraneous and represent the source of the discrepancy. We therefore prefer to adopt Longuet-Higgins' definition of the mass-transport velocity and hence take (3.2) and (3.5) as being valid under the corresponding assumptions for an inviscid fluid.

On the basis of sinusoidal wave theory, U_M' at the boundary-layer edge decreases from the shallow-wave limit of 0·3125 as the wave depth parameter kd increases, and does so appreciably when kd is greater than about 0·3. Considerations based on cnoidal theory do not reproduce this trend and in fact show that, as kd increases beyond the shallow-water range, U_M' increases to the shallow sinusoidal wave approximation and so becomes unrealistic. Consequently comparisons of the present theory with experiment would not be much use unless the data pertained to shallow-water waves.

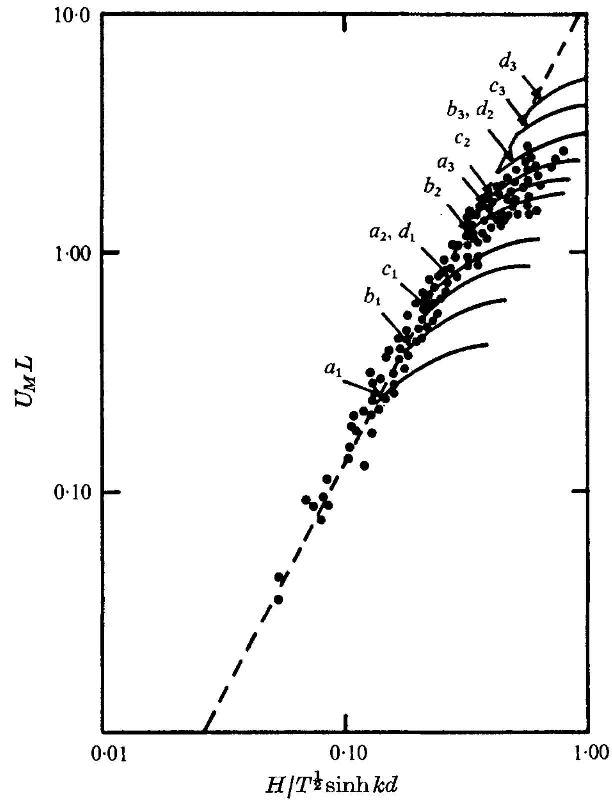

FIGURE 3. Variation of $U_M L$ with $H/T^{\frac{1}{2}}\sinh kd$. Cnoidal wave theory (solid curves): (*a*) $d = 0{\cdot}75$ ft, (*b*) $d = 1{\cdot}00$ ft, (*c*) $d = 1{\cdot}25$ ft, (*d*) $d = 1{\cdot}50$ ft; subscripts 1, 2, 3 denote $\epsilon = 0{\cdot}1,\ 0{\cdot}2,\ 0{\cdot}3$ respectively. – – –, Longuet-Higgins (1953); ●, experimental data of Brebner & Collins (1961).

Allen & Gibson (1959) have presented experimental results for waves extending to the shallow-water region and from their paper it is possible to obtain corresponding values of U'_M and κ for each experimental point, although the error in estimating κ may be appreciable. An iterative procedure based on Laitone's (1960) wave theory has been used for those few points with values of kd less than 0·38, and the calculated values of U'_M and κ, together with indications of the range of error in each value of κ, are given in figure 2. Although there is some departure from the theoretical curve for cnoidal waves, the points distinctly show the predicted decrease in U'_M as κ approaches unity, a trend which is not accounted for by sinusoidal wave theory.

Further measurements of the bottom mass-transport velocity for waves extending to the shallow-water range have been reported by Brebner & Collins (1961; see also Collins 1963). They found the bottom mass transport to become progressively smaller than Longuet-Higgins' (1953) result as the group

$$H/T^{\frac{1}{2}}\sinh kd$$

(where $T = 2\pi/\omega$) increases beyond a certain value, and attributed this to the onset of turbulence in the boundary layer at a Reynolds number based on δ of 160. However, this conclusion has since been questioned (e.g. by Sleath 1974) and the flow is now thought to remain laminar up to a Reynolds number of about $400 \times 2^{\frac{1}{2}}$.

Unfortunately Brebner & Collins' (1961) published data cannot be reduced to the form given here, but it is possible on the basis of cnoidal theory to transform the result (2.37) into an implicit relationship between the two dimensional groups they used, $U_M L$ (where $L = 2\pi/k$) and $H/T^{\frac{1}{2}} \sinh kd$. This has been carried out for the values of d they specified and typical values of ϵ, and the resultant curves are compared with Brebner & Collins' (1961) data in figure 3.

The cnoidal wave solution is not particularly suitable for such a plot and the curves have been terminated where they predict larger values of $U_M L$ than does Longuet-Higgins' (1953) theory. When kd increases beyond this limit, the present theory breaks down and, as mentioned previously, does not reproduce the reduction in U'_M that is predicted by sinusoidal theory. On the other hand Longuet-Higgins (1953, p. 572) has pointed out that for small values of kd his results may not hold unless, essentially, ϵ is very small, and consequently over this range the results based on cnoidal theory are probably more representative.

Although the calculated curves are based only on typical values of ϵ, it is apparent that for the depths indicated they show a reduction in $U_M L$ from Longuet-Higgins' (1953) solution beginning in approximately the same range of $H/T^{\frac{1}{2}} \sinh kd$ as that in the experimental results. Indeed, regardless of the probable extent of experimental error, since values of ϵ and d are uncorrelated, the points follow the trend of the curves based on cnoidal theory as well as can be expected. The behaviour found by Brebner & Collins (1961) to occur at the higher values of $H/T^{\frac{1}{2}} \sinh kd$ is, then, in accordance with the present theory and is not necessarily due to transition to turbulence.

This study was carried out while the author held a National Research Council Postdoctoral Research Associate appointment at the Joint Tsunami Research Effort, Environmental Research Laboratories, NOAA, Honolulu, Hawaii, and he is grateful to the National Research Council for its support.

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